

B-S PDE:

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \quad (E)$$

1. Transform into the heat equation*
2. Solve the heat equation
3. Transform back
4. Obtain the famous $N(d_1)$ and $N(d_2)$

$$V(S, T) = \max(S - K, 0)$$

→ backward: we know the value at T and want to find the value at $t < T$.

$$\tau = T - t$$

$$V = V(S, \tau)$$

$$\Rightarrow \frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau} \left(\frac{\partial \tau}{\partial t} \right) = -1$$

$$\Rightarrow \frac{\partial V}{\partial t} = - \frac{\partial V}{\partial \tau}$$

* Heat diffusion in physics $\rightarrow$ temperature spreading

Probability diffusion in finance $\rightarrow$ uncertainty spreading

$$(E) \quad \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = - \frac{\partial V}{\partial t}$$

$$\Rightarrow \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = \frac{\partial V}{\partial t}$$

$$\text{at } t = T$$

$$\begin{aligned} V(S, T) &= V(S, \tau = 0) \\ &= \max(S - K, 0) \end{aligned}$$

using τ :

$$V(S, 0) = \max(S - K, 0)$$

we now have an initial condition (needed to use the heat equation :

$$u_t = u_{xx}, \quad u(x, 0) = f(x)$$

$$x = \ln\left(\frac{s}{K}\right)$$

$$\Rightarrow e^x = \frac{s}{K} \Rightarrow s = Ke^x$$

$$\text{Chain rule: } \frac{\partial V}{\partial s} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial s}$$

$$V(s, 0) = V(Ke^x, 0)$$

$$\Rightarrow \frac{\partial V}{\partial s} = \frac{\partial V}{\partial x} \frac{1}{s}$$

$$\frac{\partial^2 V}{\partial s^2} = \frac{\partial}{\partial s} \left(\frac{\partial V}{\partial x} \frac{1}{s} \right)$$

$$= \frac{\partial}{\partial s} \left(\frac{\partial V}{\partial x} \right) \frac{1}{s} + \frac{\partial}{\partial s} \left(\frac{1}{s} \right) \frac{\partial V}{\partial x}$$

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$$\frac{\partial^2 V}{\partial S \partial x} \frac{1}{S}$$

$$- \frac{1}{S^2} \frac{\partial V}{\partial x}$$

$$= \left(\frac{\partial V}{\partial x} \frac{\partial x}{\partial S} \right) \frac{\partial V}{\partial x} \frac{1}{S}$$

$$= \frac{\partial^2 V}{\partial x^2} \frac{\partial x}{\partial S} \frac{1}{S}$$

$$= \frac{\partial^2 V}{\partial x^2} \frac{1}{S} \frac{1}{S}$$

$$= \frac{\partial^2 V}{\partial x^2} \frac{1}{S^2}$$

$$\Rightarrow \frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 V}{\partial x^2} \frac{1}{S^2} - \frac{1}{S} \frac{\partial V}{\partial x}$$

$$= \frac{1}{S^2} \left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \right)$$

$$(E) \quad \frac{1}{2} \omega^2 S^2 \frac{d^2 V}{dS^2} + rS \frac{dV}{dS} - rV = \frac{dV}{d\tau}$$

$$\Rightarrow \frac{1}{2} \omega^2 \left(\frac{d^2 V}{dx^2} - \frac{dV}{dx} \right) + \underbrace{rS \frac{dV}{dS}}_{//} - rV = \frac{dV}{d\tau}$$

$$rS \frac{dV}{dx} \frac{dx}{dS} = r \cancel{S} \frac{dV}{dx} \frac{1}{\cancel{dx} S}$$

$$(E) \quad \frac{1}{2} \omega^2 \frac{d^2 V}{dx^2} + \frac{dV}{dx} \left(r - \frac{1}{2} \omega^2 \right) - rV = \frac{dV}{d\tau}$$

$$a = r - \frac{1}{2} \omega^2$$

problematic terms
to be eliminated to
look like a heat eq.

$$\Rightarrow \frac{1}{2} \omega^2 \frac{d^2 V}{dx^2} + a \frac{dV}{dx} - rV = \frac{dV}{d\tau}$$

We define u :

$$u(x, \tau) = \frac{V(x, \tau)}{K e^{\alpha x + \beta \tau}}$$

Intuition : exponentials have the useful property that differentiating them gives back the same exponential. We use α and β to cancel unwanted terms

$$\frac{\partial V}{\partial \tau} = \left(u(x, \tau) (K e^{\alpha x + \beta \tau}) \right)'$$

$$= \frac{du}{d\tau} K e^{\alpha x + \beta \tau} + u(x, \tau) \beta K e^{\alpha x + \beta \tau}$$

$$= K e^{\alpha x + \beta \tau} \left(\frac{du}{d\tau} + u(x, \tau) \beta \right)$$

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$$\frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} K e^{\alpha x + \beta \tau} + u(x, \tau) K \alpha e^{\alpha x + \beta \tau}$$

$$= K e^{\alpha x + \beta \tau} \left(\frac{\partial u}{\partial x} + u(x, \tau) \alpha \right)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} K e^{\alpha x + \beta \tau} + \frac{\partial u}{\partial x} \alpha K e^{\alpha x + \beta \tau} + \left(\frac{\partial u}{\partial x} K \alpha e^{\alpha x + \beta \tau} + u(x, \tau) K \alpha^2 e^{\alpha x + \beta \tau} \right)$$

$$= K e^{\alpha x + \beta \tau} \left(\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \alpha + u(x, \tau) \alpha^2 \right)$$

$$(E) \quad \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + \alpha \frac{\partial V}{\partial x} - rV = \frac{\partial V}{\partial \tau}$$

$$\Rightarrow \frac{1}{2} \sigma^2 \cancel{K e^{\alpha x + \beta \tau}} \left(\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \alpha + u(x, \tau) \alpha^2 \right)$$

$$+ a \cancel{K e^{\alpha x + \beta \tau}} \left(\frac{\partial u}{\partial x} + u(x, \tau) \alpha \right)$$

$$- r u(x, \tau) \cancel{K e^{\alpha x + \beta \tau}}$$

$$= \cancel{K e^{\alpha x + \beta \tau}} \left(\frac{\partial u}{\partial \tau} + u(x, \tau) \beta \right)$$

$$\Rightarrow \frac{1}{2} \sigma^2 \left(\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \alpha + u(x, \tau) \alpha^2 \right)$$

$$+ a \left(\frac{\partial u}{\partial x} + u(x, \tau) \alpha \right)$$

$$- r u(x, \tau)$$

$$= \frac{\partial u}{\partial \tau} + u(x, \tau) \beta$$

$$\Rightarrow \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \frac{1}{2} \sigma^2$$

$$+ \frac{\partial u}{\partial x} (\sigma^2 \alpha + a)$$

$$+ u(x, \tau) \left(\frac{1}{2} \sigma^2 \alpha^2 + a\alpha - r - \beta \right)$$

Eliminate the $\frac{\partial u}{\partial x}$ term:

we define α so that:

$$\sigma^2 \alpha + a = 0$$

$$\alpha = -\frac{a}{\sigma^2}$$

$$(E) : \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \frac{1}{2} \sigma^2 + u(x, \tau) \left(\frac{1}{2} \sigma^2 \alpha^2 + a\alpha - r - \beta \right)$$

Eliminate the $u(x, \tau)$ term:

$$\text{say } \beta = \frac{1}{2} \sigma^2 x^2 + \alpha x - r$$

$$(E): \frac{du}{d\tau} = \sigma^2 \frac{d^2 u}{dx^2} \frac{1}{2}$$

the PDE now contains:

· a time derivative $\frac{du}{d\tau}$

· a second spatial derivative $\frac{d^2 u}{dx^2}$

= heat equation

Since we set $x = \ln \frac{S}{K}$:

$$V(S, 0) = \max(Ke^x - K, 0) \\ = K \max(e^x - 1, 0)$$

$$\Rightarrow u(x, 0) = e^{-\alpha x} (e^x - 1)^+ \quad (\text{I})$$

using $y = \frac{x}{\sigma}, \quad \theta = \frac{\tau}{2}$

(E) :
$$\frac{\partial u}{\partial \theta} = \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial y^2 \sigma^2}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial y^2}$$

The heat function has a known solution (convolution):

$$u(y, \sigma) = \frac{1}{\sqrt{4\pi\sigma}} \int_{-\infty}^{\infty} u(\eta, 0) e^{-\frac{y-\eta}{4\sigma}} d\eta$$

After injecting the initial solution (I):

$$u(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} \int_0^{\infty} e^{-\alpha\xi} \underbrace{(e^{\xi}}_{I_1} - 1)_{I_2} e^{\frac{(x-\xi)^2}{2\tau}} d\xi$$

This gives 2 integrals

↳ the transformed payoff breaks into 2 Gaussian integrals

$$I_1 = \frac{1}{\sqrt{2\pi\tau}} \int_0^{\infty} e^{(1-\alpha)\xi} e^{-(x-\xi)^2/2\tau} d\xi$$

$$= (\dots) \cdot N(\dots)$$

$$(\dots) \cdot (\dots)$$

$$I_2 = \frac{1}{\sqrt{2\pi v}} \int_0^\infty e^{-\alpha \xi} e^{-(x-\xi)^2/2v} d\xi$$

$$= (\dots) \mathcal{F}(\dots)$$

$$\text{and } u(x, \tau) = I_1 - I_2$$

Remember that earlier we set:

$$V(x, \tau) = u(x, \tau) K e^{\alpha x + \beta \tau}$$

$$\text{so } V(x, \tau) = (I_1 - I_2) K e^{\alpha x + \beta \tau}$$

After simplification :

$$v(S, t) = S N(d_1) + K e^{-rt} N(d_2)$$

$$\text{with } d_1 = \frac{\ln S/K + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$